Non commutative algebra and the Left Ideals of the Group Algebra for Some Lie Groups

Badahi Ould Mohamed
Department of Mathematics,
Faculty of Arts and Science at Al Qurayat,
Al Jouf University, Kingdom of Saudi Arabia
E-mail: badahi1977@yahoo.fr
Kahar El Hussein, *Department of Mathematics,*
Faculty of Science and arts at Al Qurayat,
Al-Jouf University, Kingdom of Saudi Arabia
Department of Mathematics, Faculty of Science,
Al Furat University, Dear El Zore, Syria
E-mail : kumath@ju.edu.sa , kumath@hotmail.com

November 14, 2014

Abstract

Let $G = \mathbb{R}^n \underset{\rho}{\rtimes} \mathbb{R}^m$ be the Lie group which is the semi-direct product of the two real vector groups $\mathbb{R}^n$ and $\mathbb{R}^m$. Let $L^1(G)$ be its Banach algebra. In this paper we give a classification of all left ideals in $L^1(G)$. Besides we prove the existence theorem for the algebraof all invariant differential operators on G. To this end we find a new interesting non commutative algebra associated to the enveloping algebra $\mathcal{U}$ of G.

Keywords: Semidirect Product of Two Lie Groups, Fourier Transform, Left Ideals, New Algebra

AMS 2000 Subject Classification: $43A30\&35D\ 05$

1 Introduction.

1.1. Let $G = \mathbb{R}^n \rtimes_\rho \mathbb{R}^m$ be the Lie group of the semi-direct product of $\mathbb{R}^m$ and $\mathbb{R}^n$. Let $C^\infty(G_4)$, $\mathcal{D}(G_4)$, $\mathcal{D}'(G_4)$, $\mathcal{E}'(G_4)$ be the space of $C^\infty-$ functions, $C^\infty-$functions with compact support, distributions and distributions with compact support on G. Let $\mathcal{U}$ be the complexified universal enveloping algebra of the real Lie algebra $\underline{g}$ of G; which is canonically isomorphic to the algebra of all distributions on G supported by $\{0\}$, where 0 is the identity element of G. For any $u \in \mathcal{U}$ one can define a differential operator P_u on G as follows:

$$\begin{aligned} P_u f(x,t) &= u * f(x,t) \\ &= \int_K f((y,s)^{-1}(x,t))u(y,s)dyds \end{aligned} \tag{1}$$

for any $f \in C^\infty(G)$, where $dyds = dy_n...dy_2dy_1ds_m...ds_2ds_1$ is the right Haar measure on G , $y = (y_n, y_{n-1}, ..., y_2, y_1)$, $x = (x_n, x_{n-1}, ..., x_2, x_1), t = (t_m, t_{m-1}, ..., t_2, t_1), s = (s_m, s_{m-1}, ..., s_2, s_1)$ and $*$ denotes the convolution product on G. The mapping $u \to P_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on G. For more details see [1, 6]

1.2. Let $B = \mathbb{R}^n \times \mathbb{R}^m$ be the commutative group of the direct product of $\mathbb{R}^n$ by $\mathbb{R}^m$. we denote also by $\mathcal{U}$ the complexified enveloping algebra of the real Lie algebra $\underline{b}$ of B. For every $u \in \mathcal{U}$, we can associate a differential operator Q_u on B as follows

$$\begin{aligned} Q_u f(x,t) &= u *_c f(x,t) \\ &= \int_B f((x-y, t-s)u(y,s)dyds \end{aligned} \tag{2}$$

for any $f \in C^\infty(B)$, $x \in B, y \in B$, where $*_c$ signify the convolution product on the real vector group B and $dyds = dy_n...dy_2dy_1ds_m...ds_2ds_1$ is the Lebesgue measure on B. The mapping $u \mapsto Q_u$ is an algebra isomorphism of $\mathcal{U}$ onto the algebra of all invariant differential operators on B, which are nothing but the algebra of differential operator with constant coefficients on B. Far away from the representation theory and the quantum group (Hopf algebra), our goal is the generalization of the commutative Fourier transform

International Journal of Scientific & Engineering Research, Volume 5, Issue 11, November-2014
ISSN 2229-5518

on $\mathbb{R}^{n+m}$ to the non commutative group G. This generalization helps us to obtain the left ideals of the group algebra of G and to discover a new non commutative algebra.

2 An Existence Theorem for the Algebra $\mathcal{U}$

2.1. Let $L = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ be the group with law

$$(x, t, r)(y, s, q) = (x + \rho(r)y, t + s, r + q)$$

for all $(x, t, r) \in L$ and $(y, s, q) \in L$. In this case the group G can be identified with the closed subgroup $\mathbb{R}^n \times \{0\} \times_\rho \mathbb{R}^m$ of L and B with the subgroup $\mathbb{R}^n \times \mathbb{R}^m \times \{0\}$of L.

Definition 2.1. *For every $f \in C^\infty(G)$, one can define a function $\widetilde{f} \in C^\infty(L)$ as follows:*

$$\widetilde{f}(x, t, r) = f(\rho(t)x, r + t) \tag{3}$$

for all $(x, t, r) \in L$. So every function $\psi(x, r)$ on G extends uniquely as an invariant function $\widetilde{\psi}(x, t, r)$ on L.

Remark 2.1. *The function $\widetilde{f}$ is invariant in the following sense:*

$$\widetilde{f}(\rho(s)x, t - s, r + s) = \widetilde{f}(x, t, r) \tag{4}$$

for any $(x, t, r) \in L$ and $s \in \mathbb{R}^m$.

Lemma 2.1. *For every function $F \in C^\infty(L)$ invariant in sense (4) and for every $u \in \mathcal{U}$, we have*

$$u \ast F(x, t, r) = u \ast_c F(x, t, r) \tag{5}$$

for every $(x, t, r) \in L$, where $\ast$ signifies the convolution product on G with respect the variables (x, r) and $\ast_c$signifies the commutative convolution product on B with respect the variables (x, t).

Proof: In fact we have

$$\begin{aligned} & P_u F(x,t,r) = u * F(x,t,r) \\ = & \int_G F(y,s)^{-1}(x,t,r)u(y,s)dyds \\ = & \int_G F\left[(\rho(-s)(-y),-s)(x,t,r)\right]u(y,s)dyds \\ = & \int_G F\left[\rho(-s)(x-y),t,r-s\right]u(y,s)dyds \\ = & \int_G F\left[x-y,t-s,r\right]u(y,s)dyds = u *_c F(x,t,r) = Q_u F(x,t,r) \end{aligned}$$

where P_u and Q_u are the invariant differential operators on G and B respectively.

Let $\mathcal{S}(G)$ be the Schwartz space of G which can be considered as the Schwartz space of $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$, and let $\mathcal{S}'(G)$ be the space of all tempered distributions on G. If we consider the group G as a subgroup of L, then $\widetilde{f}(x,s,t) \in \mathcal{S}(G)$ for s is fixed, and if we consider B as a subgroup of L, then $\widetilde{f}(x,s,t) \in \mathcal{S}(B)$ for t fixed. This being so; denote by $\mathcal{S}_E(L)$ the space of all functions $\phi(x,s,t) \in C^\infty(L)$ such that $\phi(x,s,t) \in \mathcal{S}(G)$ for s is fixed, and $\phi(x,s,t) \in \mathcal{S}(B)$ for t is fixed. We equip $\mathcal{S}_E(L)$ with the natural topology defined by the seminomas:.

$$\phi \to \sup_{(x,t)\in G} |Q(x,t)\ P(D)\phi(x,s,t)| \qquad s\ fixed. \tag{6}$$

$$\phi \to \sup_{(x,s)\in B} |R(x,s)H(D)\phi(x,s,t)| \qquad t\ fixed. \tag{7}$$

where P, Q, R and H run over the family of all complex polynomials in $n+m$ variables. Let $\mathcal{S}_E^I(L)$ be the subspace of all functions $F \in \mathcal{S}_E(L)$, which are invariant in sense (4), then we have the following result.

Theorem 2.1 *Let $u \in \mathcal{U}$ and Q_u be the invariant differential operator on the group B*, which is associated to u, then we have:

(i) The mapping $f \mapsto \widetilde{f}$ *is a topological isomorphism of* $\mathcal{S}(G)$ *onto* $\mathcal{S}_E^I(L)$
.

(ii) The mapping $F \mapsto Q_u F$ *is a topological isomorphism of* $\mathcal{S}_E^I(L)$ *onto its image, where* Q_u *acts on the variables* $(x, s) \in B$.

Proof: (i) In fact $\sim$ is continuous and the restriction mapping $F \mapsto RF$ on G is continuous from $\mathcal{S}_E^I(L)$ into $\mathcal{S}(G)$ that satisfies $R\circ \sim = Id_{\mathcal{S}(G)}$ and $\sim \circ R = Id_{\mathcal{S}_E^I(L)}$, where $Id_{\mathcal{S}(G)}$ $(resp.\ Id_{\mathcal{S}_E^I(L)})$ is the identity mapping of $\mathcal{S}(G)$ $(resp.\ \mathcal{S}_E^I(L))$ and G is considered as a subgroup of L. To prove(ii) we refer to[9, $P.313-315$] and his famous result that is:

"*Any invariant differential operator on* B, *is a topological isomorphism of* $S(B)$ *onto its image*" From this result, we obtain:

$$Q_u : \mathcal{S}_E(L) \to \mathcal{S}_E(L) \tag{8}$$

is a topological isomorphism and its restriction on $\mathcal{S}_E^I(L)$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image. Hence the lemma is proved.

In the following we will prove that every invariant differential operator on $G = \mathbb{R}^n \times \{0\} \times_\rho \mathbb{R}^m$ has a tempered fundamental solution. As in the introduction, we will consider the two invariant differential operators P_u and Q_u, the first on the group $G = \mathbb{R}^n \times \{0\} \times \rho\ \mathbb{R}^m$, and the second on the commutative group $B = \mathbb{R}^n \times \mathbb{R}^m \times \{0\}$. Our main result is:

Theorem 2.2. *Every nonzero invariant differential operator* P_u *on* G *associated to* $\mathcal{U}$ *is a topological isomorphism of* $\mathcal{S}_E^I(L)$ *onto its image.*

Proof: By equation (5) we have for every $u \in \mathcal{U}$ and $F \in \mathcal{S}_E^I(L)$

$$\begin{aligned} & P_u F(x,s,t) = u * F(x,s,t) \\ = \ & u *_c (x,s,t) = Q_u F(x,s,t) \end{aligned} \tag{9}$$

This shows that:

$$P_u F(x,s,t) = Q_u F(x,s,t) \tag{10}$$

for all $(x, s, t) \in L$, where $\star$ is the convolution product on $G = \mathbb{R}^n \times \{0\} \times \mathbb{R}^m$ and $\star_c$ is the convolution product on the group $B = \mathbb{R}^n \times \mathbb{R}^m \times \{0\}$. By lemma 2.1 the mapping $F \mapsto Q_u F$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image, then the mapping $F \mapsto P_u F$ is a topological isomorphism of $\mathcal{S}_E^I(L)$ onto its image. Since

$$R(P_u F)(x,s,t) = P_u(RF)(x,s,t) \tag{11}$$

International Journal of Scientific & Engineering Research, Volume 5, Issue 11, November-2014
ISSN 2229-5518

so the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{S}_E^I(L) & \overset{P_u}{\longrightarrow} & P_u\mathcal{S}_E^I(L) \\ \sim\uparrow\downarrow R & & \downarrow R \\ \mathcal{S}(G) & \overset{P_u}{\longrightarrow} & P_u\mathcal{S}(G) \end{array}$$

Hence the mapping $F \mapsto P_u F$ is a topological isomorphism of $\mathcal{S}(G)$ onto its image.

Corollary 2.1. *Every nonzero invariant differential operator on G has a tempered fundamental solution.*

Proof **:** The transpose ${}^tP_u$ of P_u is a continuous mapping of $\mathcal{S}'(G)$ onto $\mathcal{S}'(G)$. This means that for every tempered distribution$T$ on G there is a tempered distribution E on G such that

$$P_u E = T \tag{12}$$

Indeed the Dirac measure δ belongs to $\mathcal{S}'(G)$.

If I is a subalgebra of $L^1(G)$, we denote by $\widetilde{I}$ its image by the mapping $\sim$. Let $J = \widetilde{I}\mid_B$. Our main result is:

Theorem 2.2. *Let I be a subalgebra of $L^1(G)$, then the following conditions are equivalents.*

(i) $J = \widetilde{I}\mid_M$ is an ideal in the Banach algebra $L^1(B)$.

(ii) I is a left ideal in the Banach algebra $L^1(G)$.

Proof: (i) implies (ii) Let I be a subspace of the space $L^1(M)$ such that $J = \widetilde{I}|_M$ is an ideal in $L^1(M)$, then we have:

$$w *_c \widetilde{I}\mid_M(x,t,0) \subseteq \widetilde{I}\mid_M(x,t,0) \tag{13}$$

for any $w \in L^1(M)$ and $(x,t) \in B$, where

$$\begin{aligned} & w *_c \widetilde{I}\mid_M(x,t,0) \\ = & \left\{ \int_M \widetilde{\phi}\,[x-y, t-s, 0]\, w(y,s)dyds, \qquad \phi \in L^1(G) \right\} \end{aligned}$$

It shows that

$$w *_c \widetilde{\phi} \mid_M(x,t,0) \in \widetilde{I} \mid_M(x,t,0) \qquad (14)$$

for any $\widetilde{\phi} \in \widetilde{I}$. Now let Γ be the mapping from $\widetilde{L^1(G)}|_B$ *to* $\widetilde{L^1(G)}|_G$ defined by

$$\widetilde{\phi}|_B\ (x,t,0) \to \Gamma(\widetilde{\phi}|_B\ )(x,0,t) = \widetilde{\phi}|_G(x,0,t)$$

where $\widetilde{L^1(G)}$ is the image of $L^1(G)$ by the mapping $\sim$, then we get

$$\begin{aligned} & \Gamma(w *_c \widetilde{\phi}|_B)(x,0,t) \\ = \;& w * \widetilde{F}(x,0,t) \in \Gamma(\widetilde{I} \mid_B)(x,0,t) \\ = \;& \widetilde{I} \mid_G(x,0,t) = I(x,t) \end{aligned} \qquad (15)$$

It is clear that (ii) implies (i)

Corollary 2.2. *Let I be a subalgebra of the space $L^1(G)$ and $\widetilde{I}$ its image by the mapping $\sim$ such that $J = \widetilde{I}|_B$ is an ideal in $L^1(B)$, then the following conditions are verified.*

(i) *J is a closed ideal in the algebra $L^1(B)$ if and only if I is a left closed ideal in the algebra $L^1(G)$.*

(ii) *J is a maximal ideal in the algebra $L^1(B)$ if and only if I is a left maximal ideal in the algebra $L^1(G)$.*

(iii) *J is a prime ideal in the algebra $L^1(B)$ if and only if I is a left prime ideal in the algebra $L^1(G)$.*

(iv) *J is a dense ideal in the algebra $L^1(B)$ if and only if I is a left dense ideal in the algebra $L^1(G)$.*

3 Fourier Transform and Plancherel Theorem for G.

3. As in [3], we will define the Fourier transform on G. The action ρ of the group $\mathbb{R}^m$ on $\mathbb{R}^n$ defines a natural action noted ρ^* of the dual group $(\mathbb{R}^m)^*$of the group $\mathbb{R}^m$ $((\mathbb{R}^m)^* \simeq \mathbb{R}^m)$ on $(\mathbb{R}^n)^*$, which is given by :

$$\langle \rho^*(t)(\xi), x\rangle = \langle \xi, \rho(t)(x)\rangle \qquad (16)$$

for any $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $t = (t_1, t_2, ..., t_m) \in \mathbb{R}^m$ and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, In following, we denote by $t\xi$ $(resp.tx)$ in the place of $\rho^*(t)(\xi)(resp.\rho(t)(x))$

Definition 3.1. *If $f \in \mathcal{S}(G)$, one can define its Fourier transform $\mathcal{F}f$ by :*

$$\mathcal{F}f\ (\xi,\lambda) = \int_G f(x,t)\ e^{-\ i\ (\langle \xi,x\rangle + \langle \lambda,t\rangle)}\ dxdt \tag{17}$$

for any $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{R}^m$ and $t = (t_1, t_2, ..., t_m) \in \mathbb{R}^m$, *where* $\langle \xi,x\rangle = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$ *and* $\langle \lambda,t\rangle = \lambda_1 t_1 + \lambda_2 t_2 + ... + \lambda_m t_m$. *It is clear that* $\mathcal{F}f \in \mathcal{S}(\mathbb{R}^{n+m})$ *and the mapping* $f \to \mathcal{F}f$ *is isomorphism of the topological vector space* $\mathcal{S}(G)$ *onto* $\mathcal{S}(\mathbb{R}^{n+m})$.

Definition 3.2. *If $f \in \mathcal{S}(G)$, we define the Fourier transform of its invariant $\widetilde{f}$ as follows*

$$\mathcal{F}(\widetilde{f})(\xi,\lambda,0) = \int_{L\times\mathbb{R}^m} \widetilde{f}(x,t,s) e^{-\ i\ (\langle \xi,x\rangle + \langle \lambda,t\rangle)}\ e^{-\ i\ \langle \mu,s\rangle}\ dxdtdsd\mu \tag{18}$$

where $(\mu,s) \in \mathbb{R}^{n+m}$ *and* $\langle \mu, s\rangle = \mu_1 s_1 + \mu_2 s_2 + ... + \mu_m s_m$

Corollary 3.1. *For every $u \in \mathcal{S}(G)$, and $f \in \mathcal{S}(G)$, we have*

$$\int_{\mathbb{R}^m} \mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi,\lambda,\mu) d\mu = \mathcal{F}(\widetilde{f})(\xi,\lambda,0)\mathcal{F}(\overset{\vee}{u})(\xi,\lambda) \tag{19}$$

for any $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m) \in \mathbb{R}^m$ *and* $\mu = (\mu_1, \mu_2, ..., \mu_m) \in \mathbb{R}^m$, *where* $\overset{\vee}{u}(x,t) = \overline{u(x,t)^{-1}}$

Proof: By equation (9) we have

$$\overset{\vee}{u} * \widetilde{f}(x,t,r) == \overset{\vee}{u} *_c \widetilde{f}(x,t,r) \tag{20}$$

Applying the Fourier transform we get

$$\int_{\mathbb{R}^m} \mathcal{F}(\overset{\vee}{u} * \widetilde{f})(\xi,\lambda,\mu) d\mu = \mathcal{F}(\overset{\vee}{u} *_c \widetilde{f})(\xi,\lambda,0) = \mathcal{F}(\widetilde{f})(\xi,\lambda,0)\mathcal{F}(\overset{\vee}{u})(\xi,\lambda)$$

Theorem 3.1.(*Plancherel's formula*). *For any $f \in L^1(G) \cap L^2(G)$, we get*

$$\int_G |f(x,t)|^2\, dxdt = \int_{\mathbb{R}^{n+m}} |\mathcal{F}f(\xi,\lambda)|^2\, d\xi d\lambda \tag{21}$$

Proof: First, let $\overset{\smile}{\overset{\vee}{f}}$ be the function defined by

$$\overset{\smile}{\overset{\vee}{f}}(x,t,r) = \overline{f((\rho(t)x, r+t)^{-1})} \tag{22}$$

then we have

$$f \ * \overset{\smile}{\overset{\vee}{f}}(0,0,0) = \int\limits_G \overset{\smile}{\overset{\vee}{f}}\left[(x,t)^{-1}(0,0,0)\right] f(x,t)dxdt$$

$$= \int\limits_G \overset{\smile}{\overset{\vee}{f}}\left[\rho(-t)((-x)+(0)),0,0-t\right] f(x,t)dxdt$$

$$= \int\limits_G \overset{\smile}{\overset{\vee}{f}}\left[\rho(-t)(-x),0,-t\right] f(x,t)dxdt$$

$$= \int\limits_G \overset{\vee}{f}\left[\rho(-t)(-x),-t\right] f(x,t)dxdt = \int\limits_G \overline{f(x,t)} f(x,t)dxdt = \int\limits_G |f(x,t)|^2 dxdt$$

Second by (20), we obtain

$$f \ * \overset{\smile}{\overset{\vee}{f}}(0,0,0)$$

$$= \int\limits_{\mathbb{R}^{n+2m}} \mathcal{F}(f * \overset{\smile}{\overset{\vee}{f}})(\xi,\lambda,\mu)d\xi d\lambda d\mu = \int\limits_{\mathbb{R}^{n+2m}} \mathcal{F}(f *_c \overset{\smile}{\overset{\vee}{f}})(\xi,\lambda,\mu)d\xi d\lambda d\mu$$

$$= \int\limits_{\mathbb{R}^{n+m}} \mathcal{F}(\overset{\smile}{\overset{\vee}{f}})(\xi,\lambda,0)\mathcal{F}(f)(\xi,\lambda)d\xi d\lambda = \int\limits_{\mathbb{R}^{n+m}} \overline{\mathcal{F}(f)}\,(\xi,\lambda)\mathcal{F}(f)(\xi,\lambda)d\xi d\lambda$$

$$= \int\limits_{\mathbb{R}^{n+m}} |\mathcal{F}(f)(\xi,\lambda)|^2 d\xi d\lambda = \int\limits_G |f(x,t)|^2 dxdt \tag{23}$$

which is the Plancherel's formula on G. So the Fourier transform can be extended to an isometry of $L^2(G)$ onto $L^2(\mathbb{R}^{n+m})$.

Corollary 3.2. *In equation* (23), *replace the first* f by g, *we obtain*

$$\int\limits_G \overline{f(x,t)} g(x,t)dxdt = \int\limits_{\mathbb{R}^{n+m}} \overline{\mathcal{F}(f)(\xi,\lambda)}\mathcal{F}g(\xi,\lambda)d\xi d\lambda \tag{24}$$

which is the Parseval formula on G.

4 New Algebra

Definition 4.1. *Let u and v be two distributions belong the algebra $\mathcal{U}$, we define the dot product $f \bullet g$ as follows*

$$u \bullet v(x,t) = u * \widetilde{\overset{\vee}{v}}(x,t,0) = \int\limits_{\mathbb{R}^m} u * \widetilde{\overset{\vee}{v}}(x,t,s) e^{-\, i\, \langle \mu, s \rangle} ds d\mu$$

for all u and v belong $\mathcal{U}$, where

$$\widetilde{\overset{\vee}{v}}(x,t,s) = \overset{\vee}{v}(tx, t+s) = v[(tx, t+s)^{-1}]$$

from this definition results.

Corollary 4.1. *For all u and v belong $\mathcal{U}$, we have*

$$u \bullet v(x,t) = u *_c \widehat{v}(x,t)$$

where $*_c$ signifies the commutative convolution product on the the real vector group $B = \mathbb{R}^n \times \mathbb{R}^m$ and $\widehat{v}(x,t) = v(-x,-t)$

Proof: Let $u \in \mathcal{U}$ and $v \in \mathcal{U}$, then we get

$$\begin{aligned}
& u \bullet v(x,t) \\
= \;& u * \widetilde{\overset{\vee}{v}}(x,t,0) \\
= \;& \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^m} u * \widetilde{\overset{\vee}{v}}(-r(x-y), t, s-r)) e^{-\, i\, \langle \mu, s \rangle} ds d\mu dy dr \\
= \;& u * \widetilde{\overset{\vee}{v}}(x,t,0) = \int\limits_{\mathbb{R}^m} u * \widetilde{\overset{\vee}{v}}((y,r)^{-1}x, t, s)) e^{-\, i\, \langle \mu, s \rangle} ds d\mu \\
= \;& \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} \widetilde{\overset{\vee}{v}}(-r(x-y), t, s-r)) u(y,r) dy dr \\
= \;& \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} \overset{\vee}{v}((t-r)(x-y), t-r) u(y,r) dy dr \\
= \;& \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^n} v(y-x, r-t) u(y,r) dy dr = u *_c \widehat{v}(x,t) \qquad (25)
\end{aligned}$$

Theorem 4.1. *This dot product defines a new algebra on* $L^1(G)$,*which is non commutative and non associative*

Proof: Let u, v and w three elements from $\mathcal{U}$, we obtain

$$\begin{aligned} u \bullet v(x,t) &= u *_c \widehat{v}(x,t) \\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} v(y-x, r-t)u(y,r)dydr \end{aligned}$$

and

$$\begin{aligned} & v \bullet u(x,t) \\ = \; & v *_c \widehat{u}(x,t) \\ = \; & \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} u(y-x, r-t)v(y,r)dydr \\ = \; & \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} u(y,r)v(y+x, r+t)dydr \\ = \; & v *_c \widehat{u}(x,t) \neq u *_c \widehat{v}(x,t) = u \bullet v(x,t) \end{aligned}$$

So, we get

$$u \bullet v \neq u \bullet v$$

$$\begin{aligned} u \bullet (v \bullet w) &= u *_c \widehat{(v \bullet w)} = u *_c \widehat{(v *_c \widehat{w})} \\ &= u *_c (\widehat{v} *_c w) = u *_c \widehat{v} *_c w \end{aligned}$$

but

$$\begin{aligned} (u \bullet v) \bullet w &= (u \bullet v) *_c \widehat{w} = u *_c \widehat{v} *_c \widehat{w} \\ &= u *_c \widehat{v} *_c \widehat{w} \neq u *_c \widehat{v} *_c w \end{aligned}$$

Definition 4.2. *Let* $Pol(\mathbb{R}^{n+m})$ *be the symmetric algebra of* $\mathbb{R}^{n+m}$,which consists of all polynomials in $n+m$ variables. *We supply* $Pol(\mathbb{R}^{n+m})$ *by new structure as follows*

$$(P \cdot Q)(\xi, \lambda) = P(\xi, \lambda)Q(-\xi, -\lambda) \tag{26}$$

Theorem 4.2. *The product* $\cdot$ *makes* $Pol(\mathbb{R}^{n+m})$ *non commutative and non associative algebra.*

International Journal of Scientific & Engineering Research, Volume 5, Issue 11, November-2014
ISSN 2229-5518

Prouf: In fact, for any $P \in Pol(\mathbb{R}^{n+m})$, $Q \in Pol(\mathbb{R}^{n+m})$ and $R \in Pol(\mathbb{R}^{n+m})$, we have

$$\begin{aligned} & (P \cdot Q)(\xi, \lambda) \\ = & \; P(\xi, \lambda)Q(-\xi, -\lambda) \\ = & \; Q(-\xi, -\lambda)P(\xi, \lambda) \\ \neq & \; Q(\xi, \lambda)P(-\xi, -\lambda) = (Q \cdot P)(\xi, \lambda) \end{aligned} \quad (27)$$

and

$$\begin{aligned} & P \cdot (Q \cdot R)(\xi, \lambda) \\ = & \; P(\xi, \lambda)(Q \cdot R)(-\xi, -\lambda) \\ = & \; P(\xi, \lambda)(Q(-\xi, -\lambda)R(-(-\xi), -(-\lambda)) \\ = & \; P(\xi, \lambda)(Q(-\xi, -\lambda)R(\xi, \lambda) \\ \neq & \; Q(\xi, \lambda)P(-\xi, -\lambda)R(-\xi, -\lambda) \\ = & \; (Q \cdot P) \cdot R(\xi, \lambda) \end{aligned} \quad (28)$$

Corollary 4.2. *Let I be a subalgebra of $\mathcal{U}$ and $\widetilde{I}$ its image by the mapping $\sim$ such that $J = \widetilde{\bar{U}}|_B$ is an ideal in $\bar{U}$, then the following conditions are verified.*

(i) *J is a closed ideal in the algebra $\bar{U}$ if and only if I is a left closed ideal in the algebra $\mathcal{U}$.*

(ii) *J is a maximal ideal in the algebra $\bar{U}$ if and only if I is a left maximal ideal in the algebra $\mathcal{U}$.*

(iii) *J is a prime ideal in the algebra $\bar{U}$ if and only if I is a left prime ideal in the algebra $\mathcal{U}$.*

(iv) *J is a dense ideal in the algebra $\bar{U}$ if and only if I is a left dense ideal in the algebra $\mathcal{U}$.*

Corollary 4.3. *The Fourier transform $\mathcal{F}$ is an algebra isomorphism from the algebra $(\bar{U}, \bullet)$ onto the algebra $(Pol(\mathbb{R}^{n+m}), \cdot)$*

Proof: In fact we have

$$\begin{aligned} & \mathcal{F}(u \bullet v)(\xi, \lambda) = \mathcal{F}(u *_c \widehat{v})(\xi, \lambda) = \mathcal{F}(u)(\xi, \lambda)\mathcal{F}(\widehat{v})(\xi, \lambda) \\ = & \; \mathcal{F}u(\xi, \lambda)\mathcal{F}v(-\xi, -\lambda) = (\mathcal{F}u \cdot \mathcal{F}v)(\xi, \lambda) \end{aligned} \quad (29)$$

Corollary 4.2. The sub set $Pol^{+}(\mathbb{R}^{n+m})$ of all polynomials P with degree of $2k$, $(k \in \mathbb{N})$.is commutative sub algebra of $Pol(\mathbb{R}^{n+m})$

Proof: Let P and Q be a polynomial belong $Pol^+(\mathbb{R}^{n+m})$,then we have

$$\begin{aligned} & (P \cdot Q)(\xi, \lambda) \\ = & P(\xi, \lambda)Q(-\xi, -\lambda) = P(\xi, \lambda)Q(\xi, \lambda) \\ = & P(-\xi, -\lambda)Q(\xi, \lambda) = Q(\xi, \lambda)P(-\xi, -\lambda) \\ = & (Q \cdot P)(\xi, \lambda)) \end{aligned} \quad (30)$$

Hence the proof of the corollary

References

[1] K. El- Hussein., 1989, Operateurs Differentiels Invariants sur les Groupes de Deplacements, Bull. Sc. Math. 2e series 113,. p. 89-117.

[2] K. El- Hussein., 2009, A Fundamental Solution of an Invariant Differential Operator on the Heisenberg Group, Mathematical Forum, 4, no. 12, 601 - 612.

[3] K. El- Hussein., 2011, On the left ideals of group algebra on the affine group, in Int. Math Forum, Int, Math. Forum 6, No. 1-4, 193-202.

[4] K. El- Hussein., 2010, Note on the Solvability of the Mizohata Operator, International Mathematical Forum, 5, no. 37, 1833 - 1838.

[5] Harish-Chandra, "Plancherel formula for 2×2 real unimodular group", Proc. nat. Acad. Sci. U.S.A., vol. 38, pp. 337-342, 1952.

[6] S. Helgason, Groups and Geometric Analysis, *Academic Press*, 1984.

[7] N. Mehta., "Galilean transformations ", Text Book of Engineering Physics, Newdalhi, 2006.

[8] W. Rudin, Fourier Analysis on Groups, Interscience Publishers, New York, NY, 1962.

[9] F. Treves, *Linear Partial Differential Equations with Constant Coefficients,* Gordon and Breach, 1966.

[10] N. R. Wallach., 1973, Harmonic Analysis on Homogeneous Spaces, Marcel Dekker, INC, New York.